

# PARABOLIC CONTRACTIONS OF SEMISIMPLE LIE ALGEBRAS AND THEIR INVARIANTS

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## INTRODUCTION

The ground field  $\mathbb{k}$  is algebraically closed and  $\text{char } \mathbb{k} = 0$ . Let  $G$  be a connected semisimple algebraic group of rank  $l$ , with Lie algebra  $\mathfrak{g}$ . Motivated by some problems in Representation Theory [9, 10], E. Feigin introduced recently a very interesting contraction of  $\mathfrak{g}$  [7]. This contraction is the semi-direct product  $\tilde{\mathfrak{q}} = \mathfrak{b} \ltimes (\mathfrak{g}/\mathfrak{b})^a$ , where  $\mathfrak{b}$  is a Borel subalgebra of  $\mathfrak{g}$  and the  $\mathfrak{b}$ -module  $\mathfrak{g}/\mathfrak{b}$  is regarded as an abelian ideal in  $\tilde{\mathfrak{q}}$ . Using this contraction, Feigin also defined certain degenerations of the usual flag variety of  $G$ . This leads to numerous problems of algebraic-geometric and combinatorial nature, see [4, 6, 8]. Our intention is to look at  $\tilde{\mathfrak{q}}$  from the invariant-theoretic point of view. In [19], we proved that the ring of invariants for the adjoint or coadjoint representation of  $\tilde{\mathfrak{q}}$  is always polynomial and that the enveloping algebra,  $\mathcal{U}(\tilde{\mathfrak{q}})$ , is a free module over its centre. In this paper, we generalise Feigin's construction by replacing  $\mathfrak{b}$  with an arbitrary parabolic subalgebra  $\mathfrak{p} \subset \mathfrak{g}$ . The resulting Lie algebras are said to be *parabolic contractions* of  $\mathfrak{g}$ . For arbitrary parabolic contractions, the description of the invariants of the adjoint representation is easy and remains basically the same as for  $\mathfrak{p} = \mathfrak{b}$ , while the case of the coadjoint representation requires new techniques.

Let  $P$  be a parabolic subgroup of  $G$  with  $\text{Lie } P = \mathfrak{p}$  and  $\mathfrak{n}$  the nilpotent radical of  $\mathfrak{p}$ . Fix a Levi subgroup  $L \subset P$  and a vector space decomposition  $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{l} \oplus \mathfrak{n}_-$ , where  $\mathfrak{l} = \text{Lie } L$  and  $\mathfrak{n}_-$  is the nilpotent radical of an opposite parabolic subalgebra  $\mathfrak{p}_- = \mathfrak{l} \oplus \mathfrak{n}_-$ . Using the vector space isomorphism  $\mathfrak{g}/\mathfrak{p} \simeq \mathfrak{n}_-$ , we always regard  $\mathfrak{n}_-$  as a  $P$ -module. If  $p \in \mathfrak{p}$ ,  $\eta \in \mathfrak{n}_-$ , and  $\text{pr}_- : \mathfrak{g} \rightarrow \mathfrak{n}_-$  is the projection with kernel  $\mathfrak{p}$ , then the corresponding representation of  $\mathfrak{p}$  is given by  $(p, \eta) \mapsto p \circ \eta := \text{pr}_-([p, \eta])$ . A *parabolic contraction* of  $\mathfrak{g}$  is the semi-direct product  $\mathfrak{q} = \mathfrak{p} \ltimes (\mathfrak{g}/\mathfrak{p})^a = \mathfrak{p} \ltimes \mathfrak{n}_-^a$ , where the superscript ' $a$ ' means that the  $\mathfrak{p}$ -module  $\mathfrak{n}_-$  is regarded as an abelian ideal in  $\mathfrak{q}$ . We identify the vector spaces  $\mathfrak{g}$  and  $\mathfrak{q}$  using the decomposition  $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{n}_-$ . For  $(p, \eta), (p', \eta') \in \mathfrak{q}$ , the Lie bracket in  $\mathfrak{q}$  is defined by

$$(0.1) \quad [(p, \eta), (p', \eta')] = ([p, p'], p \circ \eta' - p' \circ \eta).$$

Set  $N_-^a = \exp(\mathfrak{n}_-^a)$  and  $Q = P \ltimes N_-^a$ . Then  $Q$  is a connected algebraic group with  $\text{Lie } Q = \mathfrak{q}$  and  $N_-^a$  is an abelian normal unipotent subgroup of  $Q$ . The exponential map  $\exp : \mathfrak{n}_-^a \rightarrow$

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$N_-^a$  is an isomorphism of varieties, and elements of  $Q$  can be written as products  $s \cdot \exp(\eta)$  with  $s \in P$  and  $\eta \in \mathfrak{n}_-$ . If  $(s, \eta) \mapsto s \cdot \eta$  is the representation of  $P$  in  $\mathfrak{n}_-$ , then the adjoint representation of  $Q$  is given by

$$(0.2) \quad \text{Ad}_Q(s \cdot \exp(\eta))(p, \eta') = (\text{Ad}(s)p, s \cdot (\eta' - p \circ \eta)).$$

In this article, we consider polynomial invariants of the adjoint and coadjoint representations of  $Q$ . In the adjoint case the answer is uniform, nice, and easy. We prove that

$$\mathbb{k}[\mathfrak{q}]^Q \simeq \mathbb{k}[\mathfrak{p}]^P \simeq \mathbb{k}[\mathfrak{l}]^L$$

and the quotient morphism  $\pi_{\mathfrak{q}} : \mathfrak{q} \rightarrow \mathfrak{q}/Q$  is equidimensional. In particular,  $\mathbb{k}[\mathfrak{q}]^Q$  is a graded polynomial algebra with  $l$  generators (see Section 2). However, the degrees of basic invariants in  $\mathbb{k}[\mathfrak{q}]^Q$  and  $\mathbb{k}[\mathfrak{g}]^G$  are not the same (unless  $L = P = G$ ).

In the coadjoint case the situation is more complicated and interesting. Our main observation is that the structure of  $\mathbb{k}[\mathfrak{q}^*]^Q$  is closely related to some properties of the centraliser  $\mathfrak{g}_e \subset \mathfrak{g}$ , where  $e \in \mathfrak{n}$  is a Richardson element associated with  $\mathfrak{p}$ . It is known that  $\mathfrak{g}_e = \mathfrak{p}_e$  and therefore the groups  $P_e \subset G_e$  have the same identity component. Let  $\mathcal{S}(\mathfrak{g}_e)$  be the symmetric algebra of  $\mathfrak{g}_e$  and  $\mathcal{S}(\mathfrak{g}_e)^{G_e}$  the subalgebra of symmetric invariants. Using an  $\mathfrak{sl}_2$ -triple containing  $e$  and the (homogeneous) basic invariants  $\mathcal{F}_1, \dots, \mathcal{F}_l$  in  $\mathcal{S}(\mathfrak{g})^G$ , one can construct certain polynomials  ${}^e\mathcal{F}_1, \dots, {}^e\mathcal{F}_l \in \mathcal{S}(\mathfrak{g}_e)^{G_e}$  (see [18] and Section 3). We prove that if  ${}^e\mathcal{F}_1, \dots, {}^e\mathcal{F}_l$  are algebraically independent and generate the algebra  $\mathcal{S}(\mathfrak{g}_e)^{P_e}$  (hence  $\mathcal{S}(\mathfrak{g}_e)^{P_e} = \mathcal{S}(\mathfrak{g}_e)^{G_e}$ ), then  $\mathbb{k}[\mathfrak{q}^*]^Q = \mathcal{S}(\mathfrak{q})^Q$  is a polynomial algebra whose free generators  $\mathcal{F}_1^\bullet, \dots, \mathcal{F}_l^\bullet$  are obtained from  $\mathcal{F}_1, \dots, \mathcal{F}_l$  via a standard contraction procedure (see Theorems 3.7 and 1.1). In this situation,  $\mathcal{F}_1^\bullet, \dots, \mathcal{F}_l^\bullet$  have also the following Kostant-like property: the differentials  $d_\xi \mathcal{F}_1, \dots, d_\xi \mathcal{F}_l$  are linearly independent ( $\xi \in \mathfrak{q}^*$ ) if and only if the orbit  $Q \cdot \xi \subset \mathfrak{q}^*$  has the maximal dimension. This relies on the theory developed by the second author in [26]. Since  $\deg \mathcal{F}_i^\bullet = \deg \mathcal{F}_i$ , we see that, unlike the case of the adjoint representation of  $\mathfrak{q}$ , the algebras  $\mathcal{S}(\mathfrak{g})^G$  and  $\mathcal{S}(\mathfrak{q})^Q$  here have the same degrees of basic invariants.

A lot of information on the algebras  $\mathcal{S}(\mathfrak{g}_e)^{G_e}$  and  $\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$  is obtained in [18], and translating some of those results in the setting of parabolic contractions yields applications of Theorem 3.7. For  $\mathfrak{g}$  of type  $\mathbf{A}_l$  or  $\mathbf{C}_l$ , all nilpotent elements  $e$  satisfy the above condition on  ${}^e\mathcal{F}_1, \dots, {}^e\mathcal{F}_l$  (see [18, Section 4]), which implies that  $\mathcal{S}(\mathfrak{q})^Q$  is a polynomial algebra for all parabolic contractions of  $\mathfrak{g} = \mathfrak{sl}_{l+1}$  or  $\mathfrak{sp}_{2l}$ . For  $\mathfrak{g}$  of type  $\mathbf{B}_l$ , the same result is obtained for a special class of parabolic contractions, see an explicit description in Theorem 4.2. We also prove that, for all Richardson elements in question, the multiset of degrees  $\{{}^e\mathcal{F}_i\}$  coincides with the multiset of degrees of basic invariants in  $\mathcal{S}(\mathfrak{l})^L$  (Proposition 4.3). This provides a full description of bi-degrees of basic invariants in  $\mathcal{S}(\mathfrak{q})^Q$ .

There are also ‘good’ Richardson orbits and parabolic contractions for all simple  $\mathfrak{g}$ . The case of regular nilpotent elements, with  $P = B$ , is covered by our previous article [19],

and here we prove that Theorem 3.7 applies to the subregular nilpotent elements and hence to the contractions associated with the minimal parabolic subalgebras (see Section 5). Although subregular nilpotent elements have some peculiarities if  $\mathfrak{g}$  is of type  $G_2$ , the resulting description appears to be the same for all simple Lie algebras. Unfortunately, there are Richardson elements  $e$  (at least for  $\mathfrak{g} = \mathfrak{so}_n$ ) such that  ${}^e\mathcal{F}_1, \dots, {}^e\mathcal{F}_l$  are algebraically dependent for any choice of  $\mathcal{F}_1, \dots, \mathcal{F}_l$  [18, Example 4.1]. This implies that  $\mathcal{F}_1^\bullet, \dots, \mathcal{F}_l^\bullet$  are also algebraically dependent, and our technique does not apply. However, this does not necessarily mean that here  $\mathcal{S}(\mathfrak{q})^Q$  cannot be a polynomial algebra.

To a great extent, article [18] was motivated by the following conjecture of Premet:

*If  $e \in \mathfrak{g}$  is a nilpotent element, then  $\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$  is a graded polynomial algebra in  $l$  variables.*

Since then, it was discovered that this conjecture is false. A counterexample, with  $\mathfrak{g}$  of type  $E_8$ , is presented in [25]. Therefore, it is a challenge to classify all nilpotent elements (orbits) such that  $\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$  is graded polynomial. We hope that theory of parabolic contractions can provide new insights on the structure of the algebras  $\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$  and  $\mathcal{S}(\mathfrak{g}_e)^{G_e}$  for Richardson elements  $e$ .

*Main notation.*

- the centraliser in  $\mathfrak{g}$  of  $x \in \mathfrak{g}$  is denoted by  $\mathfrak{g}_x$ .
- $\kappa$  is the Killing form on  $\mathfrak{g}$ .
- If  $X$  is an irreducible variety, then  $\mathbb{k}[X]$  is the algebra of regular functions and  $\mathbb{k}(X)$  is the field of rational functions on  $X$ . If  $X$  is acted upon by an algebraic group  $A$ , then  $\mathbb{k}[X]^A$  and  $\mathbb{k}(X)^A$  denote the subsets of respective  $A$ -invariant functions.
- If  $\mathbb{k}[X]^A$  is finitely generated, then  $X//A := \text{Spec}(\mathbb{k}[X]^A)$  and the *quotient morphism*  $\pi : X \rightarrow X//A$  is determined by the inclusion  $\mathbb{k}[X]^A \hookrightarrow \mathbb{k}[X]$ . If  $\mathbb{k}[X]^A$  is graded polynomial, then the elements of any set of algebraically independent homogeneous generators will be referred to as *basic invariants*.
- $\mathcal{S}^i(V)$  is the  $i$ -th symmetric power of the vector space  $V$  over  $\mathbb{k}$  and  $\mathcal{S}(V) = \bigoplus_{i \geq 0} \mathcal{S}^i(V)$  is the symmetric algebra of  $V$  over  $\mathbb{k}$ ;  $\mathbb{k}[V]_n = \mathcal{S}^n(V^*)$  and  $\mathbb{k}[V] = \mathcal{S}(V^*)$ .

## 1. CONSTRUCTING INVARIANTS FOR PARABOLIC CONTRACTIONS

The Lie algebra  $\mathfrak{q} = \mathfrak{p} \ltimes \mathfrak{n}_-^a$  is an Inönü-Wigner (= 1-parameter) contraction of  $\mathfrak{g}$ . In [19, Sect. 1], we provided a general method for constructing invariants of adjoint and coadjoint representations of such contractions from invariants of the initial Lie algebra. Here we recall the relevant notation and the method in the setting of parabolic contractions.

The words ‘Inönü-Wigner contraction’ mean that the Lie bracket in  $\mathfrak{q}$  (0.1) can be obtained in the following way. Consider the invertible linear map  $c_t : \mathfrak{g} \rightarrow \mathfrak{g}$ ,  $t \in \mathbb{k} \setminus \{0\}$ ,

such that  $c_t(p + \eta) = p + t\eta$  ( $p \in \mathfrak{p}$ ,  $\eta \in \mathfrak{n}_-$ ) and define the new bracket  $[\cdot, \cdot]_{(t)}$  on the vector space  $\mathfrak{g}$  by the rule

$$[x, y]_{(t)} := c_t^{-1}([c_t(x), c_t(y)]), \quad x, y \in \mathfrak{g}.$$

Write  $\mathfrak{g}_{(t)}$  for the corresponding Lie algebra. The operator  $(c_t)^{-1} = c_{t^{-1}} : \mathfrak{g} \rightarrow \mathfrak{g}_{(t)}$  yields an isomorphism between the Lie algebras  $\mathfrak{g} = \mathfrak{g}_{(1)}$  and  $\mathfrak{g}_{(t)}$ , hence all algebras  $\mathfrak{g}_{(t)}$  are isomorphic. It is easily seen that  $\lim_{t \rightarrow 0} \mathfrak{g}_{(t)} \simeq \mathfrak{p} \ltimes (\mathfrak{g}/\mathfrak{p})^a = \mathfrak{q}$ .

To construct invariants of the coadjoint representation of  $Q$ , we use the decomposition  $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{n}_-$  and the corresponding bi-grading

$$(1.1) \quad \mathbb{k}[\mathfrak{g}^*] = \mathcal{S}(\mathfrak{g}) = \bigoplus_{i,j \geq 0} \mathcal{S}^i(\mathfrak{p}) \otimes \mathcal{S}^j(\mathfrak{n}_-)$$

If  $\mathcal{H} \in \mathcal{S}(\mathfrak{g})$  is homogeneous (of total degree  $n$ ) then  $\mathcal{H}^\bullet$  stands for its bi-homogeneous component having the highest degree with respect to  $\mathfrak{n}_-$ . That is, if  $\mathcal{H} = \sum_{a \leq i \leq b} \mathcal{H}^{(n-i,i)}$ , where  $\mathcal{H}^{(n-i,i)} \in \mathcal{S}^{n-i}(\mathfrak{p}) \otimes \mathcal{S}^i(\mathfrak{n}_-)$  and  $\mathcal{H}^{(n-b,b)} \neq 0$ , then  $\mathcal{H}^\bullet := \mathcal{H}^{(n-b,b)}$ . In this situation, we also set  $\deg_{\mathfrak{p}}(\mathcal{H}^\bullet) = n - b$  and  $\deg_{\mathfrak{n}_-}(\mathcal{H}^\bullet) = b$ .

**Theorem 1.1** ([19, Theorem 1.1]). *If  $\mathcal{H} \in \mathcal{S}^n(\mathfrak{g})^G = \mathbb{k}[\mathfrak{g}^*]_n^G$ , then  $\mathcal{H}^\bullet \in \mathcal{S}^n(\mathfrak{q})^Q = \mathbb{k}[\mathfrak{q}^*]_n^Q$ .*

Say that  $\mathcal{H}^\bullet$  is the *highest component* of  $\mathcal{H} \in \mathbb{k}[\mathfrak{g}^*]_n^G$  (with respect to the parabolic contraction  $\mathfrak{g} \rightsquigarrow \mathfrak{q}$ ). Let  $\mathcal{L}^\bullet(\mathbb{k}[\mathfrak{g}^*]^G)$  denote the linear span of  $\{\mathcal{H}^\bullet \mid \mathcal{H} \in \mathbb{k}[\mathfrak{g}^*]^G \text{ is homogeneous}\}$ . Clearly, it is a graded algebra, and Theorem 1.1 implies that  $\mathcal{L}^\bullet(\mathbb{k}[\mathfrak{g}^*]^G) \subset \mathbb{k}[\mathfrak{q}^*]^Q$ . We say that  $\mathcal{L}^\bullet(\mathbb{k}[\mathfrak{g}^*]^G)$  is the *algebra of highest components* for  $\mathbb{k}[\mathfrak{g}^*]^G$  (relative to bi-grading (1.1)).

Invariants of the adjoint representation of  $Q$  can be constructed in a similar (“dual”) way. Set  $\mathfrak{n}_-^* := \mathfrak{p}^\perp$ , the annihilator of  $\mathfrak{p}$  in  $\mathfrak{g}^*$ . Likewise,  $\mathfrak{p}^* := (\mathfrak{n}_-)^{\perp}$ . Then  $\mathfrak{g}^* = \mathfrak{n}_-^* \oplus \mathfrak{p}^*$ . Having identified the vector spaces  $\mathfrak{g}^*$  and  $\mathfrak{q}^*$ , we play the same game with the bi-grading

$$(1.2) \quad \mathbb{k}[\mathfrak{g}] = \mathcal{S}(\mathfrak{g}^*) = \bigoplus_{i,j \geq 0} \mathcal{S}^i(\mathfrak{p}^*) \otimes \mathcal{S}^j(\mathfrak{n}_-^*)$$

and homogeneous elements of  $\mathcal{S}(\mathfrak{g}^*)^G = \mathbb{k}[\mathfrak{g}]^G$ . For  $\mathcal{H} \in \mathcal{S}^n(\mathfrak{g}^*)$ , let  $\mathcal{H}_\bullet$  denote its bi-homogeneous component relative to bi-grading (1.2) having the highest degree with respect to  $\mathfrak{p}^*$ .

**Theorem 1.2** ([19, Theorem 1.2]). *If  $\mathcal{H} \in \mathbb{k}[\mathfrak{g}]_n^G$ , then  $\mathcal{H}_\bullet \in \mathbb{k}[\mathfrak{q}]_n^Q$ .*

Likewise, one obtains the respective algebra of highest components,  $\mathcal{L}_\bullet(\mathbb{k}[\mathfrak{g}]^G)$ , which can be regarded as a graded subalgebra of  $\mathbb{k}[\mathfrak{q}]^Q$ .

Since  $\mathfrak{g}$  is semisimple, one may identify  $\mathfrak{g}$  and  $\mathfrak{g}^*$  (and hence  $\mathcal{S}(\mathfrak{g})$  and  $\mathcal{S}(\mathfrak{g}^*)$ ) as  $G$ -modules using the Killing form  $\kappa$ . Note that upon this identification  $\mathcal{S}(\mathfrak{g})$  and  $\mathcal{S}(\mathfrak{g}^*)$ , one obtains two essentially different bi-gradings, (1.1) and (1.2), of one and the same algebra. Namely, since  $\mathfrak{n}_-^* \simeq \mathfrak{n}$  and  $\mathfrak{p}^* \simeq \mathfrak{p}_-$ , these bi-gradings are determined by the decompositions  $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{n}_-$  and  $\mathfrak{g}^* = \mathfrak{g} = \mathfrak{n} \oplus \mathfrak{p}_-$ , respectively. The upshot is that, for a given

homogeneous  $G$ -invariant  $\mathcal{H}$ , there are two constructions of the “highest bi-homogeneous component”,  $\mathcal{H}^\bullet$  and  $\mathcal{H}_\bullet$ . But these highest components are determined via different bi-gradings and belong to different algebras of  $Q$ -invariants!

Since  $\mathfrak{g}$  and  $\mathfrak{q}$  (as well as  $\mathfrak{g}^*$  and  $\mathfrak{q}^*$ ) are naturally identified as vector spaces, we always think of  $\mathfrak{q}$  and  $\mathfrak{q}^*$  as vector spaces equipped with the decompositions

$$\mathfrak{q} = \mathfrak{p} \oplus \mathfrak{n}_- \quad \text{and} \quad \mathfrak{q}^* = \mathfrak{n} \oplus \mathfrak{p}_-.$$

All summands here are  $P$ -modules and in both cases, the second summand is  $Q$ -stable.

**Lemma 1.3** ([19, Lemma 1.3]). *The graded algebras  $\mathbb{k}[\mathfrak{g}^*]^G$  and  $\mathcal{L}^\bullet(\mathbb{k}[\mathfrak{g}^*]^G)$  have the same Poincaré series, i.e.,  $\dim \mathbb{k}[\mathfrak{g}^*]_n^G = \dim \mathcal{L}^\bullet(\mathbb{k}[\mathfrak{g}^*]_n^G)$  for all  $n \in \mathbb{N}$ ; and likewise for  $\mathbb{k}[\mathfrak{g}]^G$  and  $\mathcal{L}_\bullet(\mathbb{k}[\mathfrak{g}]^G)$ .*

By [17, Theorem 2.7], the algebras of invariants of the adjoint and coadjoint representations of Inönü-Wigner contractions are bi-graded. The embeddings  $\mathcal{L}^\bullet(\mathbb{k}[\mathfrak{g}^*]^G) \subset \mathbb{k}[\mathfrak{q}^*]^Q$  and  $\mathcal{L}_\bullet(\mathbb{k}[\mathfrak{g}]^G) \subset \mathbb{k}[\mathfrak{q}]^Q$  (together with Lemma 1.3) prompt the natural question whether these are equalities. We will see in Section 2 that  $\mathcal{L}_\bullet(\mathbb{k}[\mathfrak{g}]^G) \subsetneq \mathbb{k}[\mathfrak{q}]^Q$  unless  $\mathfrak{l} = \mathfrak{g}$ . Moreover, the  $Q$ -invariants  $\mathcal{H}_\bullet$  play no role in describing  $\mathbb{k}[\mathfrak{q}]^Q$ . But the situation is different for the coadjoint representation. In all cases, when we can describe the algebra  $\mathbb{k}[\mathfrak{q}^*]^Q$ , the equality  $\mathcal{L}^\bullet(\mathbb{k}[\mathfrak{g}^*]^G) = \mathbb{k}[\mathfrak{q}^*]^Q$  will be an important ingredient of the final result, see Sections 3–5. Earlier, we proved that this equality holds for  $\mathfrak{p} = \mathfrak{b}$ , see [19, Section 3]. Similar phenomena occur also for the adjoint and coadjoint representations of  $\mathbb{Z}_2$ -contractions of  $\mathfrak{g}$ , see [16].

## 2. INVARIANTS OF THE ADJOINT REPRESENTATION OF $Q$

In this section, we describe the algebra of invariants and the quotient morphism for the adjoint representation of any parabolic contraction of  $\mathfrak{g}$ .

To prove that a certain set of invariants of an algebraic group action generates the whole algebra of invariants, we use the following variation of Igusa’s lemma.

**Lemma 2.1.** *Let  $A$  be a connected algebraic group acting regularly on an irreducible affine variety  $X$ . Suppose that a finitely generated subalgebra  $S \subset \mathbb{k}[X]^A$  has the following properties:*

- (i)  $Y := \operatorname{Spec} S$  is normal;
- (ii) generic fibres of  $\pi : X \rightarrow Y$  are irreducible;
- (iii)  $\dim X - \dim Y = \max_{x \in X} \dim A \cdot x$ .
- (iv)  $\operatorname{Im}(\pi)$  contains an open subset  $\Omega$  of  $Y$  such that  $\operatorname{codim}(Y \setminus \Omega) \geq 2$ .

*Then  $S = \mathbb{k}[X]^A$ . In particular, the algebra of  $A$ -invariants is finitely generated.*

*Proof.* By property (iv),  $\pi$  is dominant and then properties (ii) and (iii) imply that

(ii') the fibres of  $\pi$  over a dense open subset of  $Y$  contain a dense  $A$ -orbit.

Then properties (i), (ii'), and (iv) constitute the assumptions of Igusa's lemma, see [11, Lemma 4] or [16, Lemma 6.1]  $\square$

*Remark 2.2.* If the group  $A$  is unipotent, then the hypotheses of Lemma 2.1 imply that generic fibres of  $\pi$  are just  $A$ -orbits.

Let  $\mathfrak{z}(\mathfrak{l})$  denote the centre of  $\mathfrak{l}$  and  $\mathfrak{z}(\mathfrak{l})_{reg} = \{x \in \mathfrak{z}(\mathfrak{l}) \mid \mathfrak{g}_x = \mathfrak{l}\}$ . Then  $\mathfrak{z}(\mathfrak{l})_{reg}$  is a dense open subset of  $\mathfrak{z}(\mathfrak{l})$ .

**Lemma 2.3.** *If  $x \in \mathfrak{z}(\mathfrak{l})_{reg}$  and  $n \in \mathfrak{n}$  is arbitrary, then  $x + n$  and  $x$  belong to the same  $\text{Ad } N$ -orbit.*

*Proof.* Clearly,  $(\text{Ad } N)x \subset t + \mathfrak{n}$  for all  $x \in \mathfrak{z}(\mathfrak{l})$ . If  $x \in \mathfrak{z}(\mathfrak{l})_{reg}$ , then  $\dim(\text{Ad } N)x = \dim \mathfrak{n}$ . It is also known that the orbits of a unipotent group acting on an affine variety are closed, see e.g. [24, p. 35]. Hence  $(\text{Ad } N)x = x + \mathfrak{n}$ .  $\square$

**Proposition 2.4.** *For any parabolic subgroup  $P$  with a Levi subgroup  $L$ , we have  $\mathbb{k}[\mathfrak{p}]^P \simeq \mathbb{k}[\mathfrak{l}]^L$ . More precisely, the isomorphism  $\mathfrak{p} // P \simeq \mathfrak{l} // L$  is induced by the projection  $\mathfrak{p} \rightarrow \mathfrak{p}/\mathfrak{n} \simeq \mathfrak{l}$ .*

*Proof.* The projection  $\tau : \mathfrak{p} \rightarrow \mathfrak{p}/\mathfrak{n} \simeq \mathfrak{l}$  is surjective and  $P$ -equivariant, and the  $N$ -action on  $\mathfrak{p}/\mathfrak{n}$  is trivial. It follows that the comorphism  $\tau^\# : \mathbb{k}[\mathfrak{l}] \rightarrow \mathbb{k}[\mathfrak{p}]$  yields an  $L$ -equivariant embedding  $\mathbb{k}[\mathfrak{l}] \hookrightarrow \mathbb{k}[\mathfrak{p}]^N$ . For  $x \in \mathfrak{z}(\mathfrak{l})_{reg}$ , it follows from Lemma 2.3 that  $\tau^{-1}(x) = (\text{Ad } N)x = x + \mathfrak{n}$ . In particular,  $\max_{x \in \mathfrak{l}} \dim(\text{Ad } N)x = \dim N$ . Since all the fibres of  $\tau$  are irreducible, Lemma 2.1 applies with  $\pi = \tau$ ,  $Y = \mathfrak{p}/\mathfrak{n} \simeq \mathfrak{l}$ , etc., and we conclude that  $\mathbb{k}[\mathfrak{l}] = \mathbb{k}[\mathfrak{p}]^N$ . Hence  $\mathbb{k}[\mathfrak{p}]^P = (\mathbb{k}[\mathfrak{p}]^N)^L \simeq \mathbb{k}[\mathfrak{l}]^L$ .  $\square$

Recall that  $Q = P \ltimes N_-^a$  and  $\mathfrak{q} = \mathfrak{p} \ltimes \mathfrak{n}_-^a$ , and our goal is to describe the algebra  $\mathbb{k}[\mathfrak{q}]^Q$ .

**Theorem 2.5.** *We have  $\mathbb{k}[\mathfrak{q}]^Q \simeq \mathbb{k}[\mathfrak{p}]^P \simeq \mathbb{k}[\mathfrak{l}]^L$ . In particular,  $\mathbb{k}[\mathfrak{q}]^Q$  is a graded polynomial algebra. Furthermore, the quotient morphism  $\pi_{\mathfrak{q}} : \mathfrak{q} \rightarrow \mathfrak{q} // Q \simeq \mathbb{k}^l$  is equidimensional.*

*Proof.* 1) In view of Proposition 2.4, we have to prove that  $\mathbb{k}[\mathfrak{q}]^Q \simeq \mathbb{k}[\mathfrak{p}]^P$ . Since  $\mathbb{k}[\mathfrak{q}]^Q = (\mathbb{k}[\mathfrak{q}]^{N_-^a})^P$ , the assertion will follow from the fact that

$$(2.1) \quad \mathbb{k}[\mathfrak{q}]^{N_-^a} \simeq \mathbb{k}[\mathfrak{p}]$$

and this isomorphism is compatible with the  $P$ -actions. To this end, consider the surjective  $Q$ -equivariant projection  $\pi : \mathfrak{q} \rightarrow \mathfrak{q}/\mathfrak{n}_-^a \simeq \mathfrak{p}$ . By Eq. (0.2), the subgroup  $N_-^a \subset Q$  acts trivially on  $\mathfrak{q}/\mathfrak{n}_-^a$ . It follows that the comorphism  $\pi^\#$  yields a  $P$ -equivariant embedding  $\mathbb{k}[\mathfrak{p}] \subset \mathbb{k}[\mathfrak{q}]^{N_-^a}$ . Again, to see that this is an equality, we use Lemma 2.1. If  $x \in \mathfrak{z}(\mathfrak{l})_{reg} \subset \mathfrak{p}$  and  $\eta \in \mathfrak{n}_-^a$ , then  $x \circ \eta \in \mathfrak{n}_-^a$  and  $x \circ \eta = 0$  if and only if  $\eta = 0$ . Therefore,  $(\text{Ad}_Q N_-^a)(x) = x + \mathfrak{n}_-^a$  (use Eq. (0.2) with  $s = 1$ ,  $\eta' = 0$ , and  $p = x$ ). This implies that  $\max_{\xi \in \mathfrak{q}} \dim N_-^a \cdot \xi =$



$\dim N$ . As all the fibres of  $\pi$  are irreducible, Lemma 2.1 applies here, and we conclude that Eq. (2.1) holds.

2) Gathering all previous descriptions, we see that  $\pi_q$  is the composition

$$q = \mathfrak{p} \ltimes \mathfrak{n}_-^a \rightarrow \mathfrak{p} \rightarrow \mathfrak{p}/\mathfrak{n} \simeq \mathfrak{l} \rightarrow \mathfrak{l}/L.$$

Since  $\pi_l : \mathfrak{l} \rightarrow \mathfrak{l}/L$  is equidimensional [13], the equidimensionality of  $\pi_q$  follows.  $\square$

Comparing with the adjoint representation of  $G$ , we see that the algebra of  $Q$ -invariants remains polynomial, but the degrees of basic invariants somehow decrease. This certainly means that here  $\mathcal{L}_\bullet(\mathbb{k}[\mathfrak{g}]^G) \subsetneq \mathbb{k}[q]^Q = \mathbb{k}[\mathfrak{l}]^L$  unless  $\mathfrak{l} = \mathfrak{g}$ .

### 3. ON INVARIANTS OF THE COADJOINT REPRESENTATION OF $Q$

In this section, we present some general properties of the invariants of the coadjoint representation of  $Q$ . This will be a base for the explicit results on  $\mathcal{S}(\mathfrak{q})^Q$  presented in Sections 4 and 5. The coadjoint representation is much more interesting than the adjoint one since  $\mathbb{k}[q^*] = \mathcal{S}(\mathfrak{q})$  is a Poisson algebra,  $\mathcal{S}(\mathfrak{q})^Q$  is the centre of this Poisson algebra, and  $\mathcal{S}(\mathfrak{q})$  is related to the enveloping algebra of  $\mathfrak{q}$  via the Poincaré-Birkhoff-Witt theorem. However, the coadjoint representation is also much more complicated, and we cannot describe  $\mathcal{S}(\mathfrak{q})^Q$  for all parabolic contraction.

Recall that  $\mathfrak{q}$  is isomorphic to  $\mathfrak{p} \oplus \mathfrak{g}/\mathfrak{p} \simeq \mathfrak{p} \oplus \mathfrak{n}_-$  as a vector space and a  $P$ -module. Then  $\mathfrak{q}^*$  is identified with the direct sum of  $P$ -modules  $\mathfrak{n} \oplus \mathfrak{p}_-$ , where  $\mathfrak{p}_- \simeq \mathfrak{p}^*$ .

By a seminal result of R.W. Richardson [22],  $P$  has a dense orbit in  $\mathfrak{n}$ ; in other words, there exists  $e \in \mathfrak{n}$  such that  $[\mathfrak{p}, e] = \mathfrak{n}$ . Then  $\mathfrak{p}$  is called a *polarisation* of  $e$ . The nilpotent elements of  $\mathfrak{g}$  occurring as representatives of dense  $P$ -orbits in  $\mathfrak{n} = \mathfrak{p}^{nil}$  for some  $P$  (and the corresponding  $G$ -orbits) are said to be *Richardson* or *polarisable*. If  $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$  is a polarisation of  $e$ , then

$$\dim G \cdot e = 2 \dim \mathfrak{n} = \dim G/L \quad \text{and} \quad \mathfrak{p}_e = \mathfrak{g}_e.$$

However, it can happen that  $P_e \subsetneq G_e$ , and in such cases  $G_e$  is necessarily disconnected. The orbit  $G \cdot e$  depends only on the Levi subalgebra of  $\mathfrak{p}$ . That is, if  $\mathfrak{p}' = \mathfrak{l} \oplus \mathfrak{n}'$  is another parabolic subalgebra with the same Levi subalgebra  $\mathfrak{l}$  and  $e' \in \mathfrak{n}'$  is Richardson, then  $G \cdot e = G \cdot e'$ .

If  $\mathfrak{g}$  is of type  $A_l$ , then all nilpotent elements are Richardson, otherwise this is not always the case. We refer to [12] for an explicit description of the Richardson elements and their polarisations for all classical Lie algebras.

For an algebraic group  $A$  with Lie algebra  $\mathfrak{a}$ , the *index* of  $\mathfrak{a}$ ,  $\text{ind } \mathfrak{a}$ , is defined as the minimal codimension of an  $A$ -orbit in the coadjoint representation. By the Rosenlicht theorem [2, 1.6], one also has  $\text{ind } \mathfrak{a} = \text{tr.deg } \mathbb{k}(\mathfrak{a}^*)^A$ . The index of a reductive Lie algebra

equals the rank. It is easily seen that the index cannot decrease under contractions, hence  $\text{ind } \mathfrak{q} \geq \text{ind } \mathfrak{g} = \text{rk } \mathfrak{g}$ .

**Theorem 3.1.** *We have  $\text{ind } \mathfrak{q} = \text{rk } \mathfrak{g}$  for any parabolic contraction.*

*Proof.* As  $\mathfrak{q} = \mathfrak{p} \ltimes \mathfrak{n}^a$  is a semi-direct product with an abelian ideal  $\mathfrak{n}^a$ , one can use Raïs' formula [21] for computing  $\text{ind } \mathfrak{q}$ . Namely,

$$\text{ind } \mathfrak{q} = \text{ind } \mathfrak{p}_\xi + (\dim \mathfrak{n} - \max_{x \in \mathfrak{n}} \dim P \cdot x),$$

where  $\mathfrak{n}$  occurs as the dual  $P$ -module to  $\mathfrak{n}^a \simeq \mathfrak{g}/\mathfrak{p}$  and  $\xi \in \mathfrak{n}$  is a generic element. Because of the existence of Richardson elements in  $\mathfrak{n}$ , Raïs' formula boils down to the equality  $\text{ind } \mathfrak{q} = \text{ind } \mathfrak{p}_e = \text{ind } \mathfrak{g}_e$ . For all Richardson elements, the equality  $\text{ind } \mathfrak{g}_e = \text{rk } \mathfrak{g}$  is proved in [5] (see also [15, Cor. 3.4] for the case of subregular elements).  $\square$

*Remark 3.2.* The assertion that “ $\text{ind } \mathfrak{g}_e = \text{rk } \mathfrak{g}$  for all nilpotent elements of  $\mathfrak{g}$ ” is known as the *Elashvili conjecture*, see [15, Sect. 3]. Despite the extreme simplicity of the statement, there is no general conceptual proof as yet. However, partial results of several authors (Yakimova, Charbonnel & Moreau) and computer computations (de Graaf) together provide an affirmative answer to the Elashvili conjecture. A historic outline can be found in [5].

**3.1. A sufficient condition for algebraic independence.** Let  $\mathcal{F}_1, \dots, \mathcal{F}_l$  be any set of basic invariants in  $\mathbb{k}[\mathfrak{g}]^G$ . A classical result of Kostant asserts that, for  $x \in \mathfrak{g}$ , we have  $\dim \mathfrak{g}_x = l$  if and only if the differentials  $d_x \mathcal{F}_1, \dots, d_x \mathcal{F}_l$  are linearly independent, see [13, Theorem 9]. As the elements  $x$  of  $\mathfrak{g}$  with  $\dim \mathfrak{g}_x = \text{rk } \mathfrak{g} = l$  are said to be regular, this assertion is sometimes called *Kostant's regularity criterion*. Having identified  $\mathbb{k}[\mathfrak{g}]$  and  $\mathcal{S}(\mathfrak{g})$ , one can rewrite this property in terms of  $\dim \mathfrak{g}_\xi$  ( $\xi \in \mathfrak{g}^*$ ) and  $\{d_\xi \mathcal{F}_i\}_{i=1}^l$  for  $\mathcal{F}_1, \dots, \mathcal{F}_l \in \mathcal{S}(\mathfrak{g})^G$ . There is also a more fancy way to express this criterion.

**Definition 1** (cf. [26, Definition 2.2]). Let  $\mathfrak{a}$  be an algebraic Lie algebra with  $\text{ind } \mathfrak{a} = l$  and  $\mathcal{F}_1, \dots, \mathcal{F}_l$  algebraically independent elements of  $\mathcal{S}(\mathfrak{a})^{\mathfrak{a}}$ . Suppose that  $d_\gamma \mathcal{F}_1, \dots, d_\gamma \mathcal{F}_l$  are linearly independent if and only if  $\dim \mathfrak{a}_\gamma = \text{ind } \mathfrak{a}$  ( $\gamma \in \mathfrak{a}^*$ ). Then we say that the polynomials  $\mathcal{F}_i$  ( $1 \leq i \leq l$ ) satisfy the *Kostant equality* (in  $\mathfrak{a}$ ).

Recall from Section 1 that we have defined a linear operator  $c_t : \mathfrak{g} \rightarrow \mathfrak{g}$  ( $t \in \mathbb{k}^\times$ ) and a family of Lie algebras  $\mathfrak{g}_{(t)}$  such that  $\lim_{t \rightarrow 0} \mathfrak{g}_{(t)} = \mathfrak{q}$ . Having extended  $c_t$  to  $\mathcal{S}(\mathfrak{g})$  in the obvious way, one can regard  $c_t(\mathcal{H})$  as an element of  $\mathcal{S}(\mathfrak{g})[t]$  for any  $\mathcal{H} \in \mathcal{S}(\mathfrak{g})$ . Then define  $\deg_t \mathcal{H}$  to be the usual degree in  $t$  of  $c_t(\mathcal{H})$ . If  $\deg_t \mathcal{H} = d$ , then the limit  $\lim_{t \rightarrow 0} t^d c_{t^{-1}}(\mathcal{H})$  exists and is a nonzero element of  $\mathcal{S}(\mathfrak{g})$ . If  $\mathcal{H}$  is homogeneous, then it follows immediately from the definition of  $c_t$  that  $\lim_{t \rightarrow 0} t^d c_{t^{-1}}(\mathcal{H}) = \mathcal{H}^\bullet$  and  $\deg_t \mathcal{H} = \deg_{\mathfrak{n}_-}(\mathcal{H}^\bullet)$ . See also [26, Sect. 3] for a more general setup.

Since  $\text{ind } \mathfrak{q} = \text{rk } \mathfrak{g}$  (Theorem 3.1), the theory developed in [26] yields the following statement in the setting of the parabolic contractions:



**Theorem 3.3** ([26, Theorem 3.8]). *If  $\mathcal{F}_1, \dots, \mathcal{F}_l \in \mathcal{S}(\mathfrak{g})^G$  are the basic invariants, then  $\sum_{i=1}^l \deg_{\mathfrak{n}_-}(\mathcal{F}_i^\bullet) \geq \dim \mathfrak{n}$ . Moreover, the equality holds if and only if  $\mathcal{F}_1^\bullet, \dots, \mathcal{F}_l^\bullet$  are algebraically independent, and in this case the polynomials  $\{\mathcal{F}_i^\bullet\}_{i=1}^l$  also satisfy the Kostant equality in  $\mathfrak{q}$ .*

In Sections 4 and 5, we consider certain parabolic contractions, where we are able to prove that  $\mathcal{F}_1^\bullet, \dots, \mathcal{F}_l^\bullet$  are algebraically independent. And this will finally lead us to the conclusion that  $\mathcal{S}(\mathfrak{q})^Q = \mathbb{K}[\mathcal{F}_1^\bullet, \dots, \mathcal{F}_l^\bullet] = \mathcal{L}^\bullet(\mathcal{S}(\mathfrak{g})^G)$  is a polynomial algebra. However, it can happen that  $\mathcal{F}_1^\bullet, \dots, \mathcal{F}_l^\bullet$  are algebraically dependent for any choice of the basic invariants  $\mathcal{F}_1, \dots, \mathcal{F}_l$  (see Remark 4.6 below).

**3.2. Symmetric invariants of centralisers and contractions.** Here we explain an astonishing relationship between invariants of  $(\mathfrak{q}, \text{ad}^*)$  with  $\mathfrak{q} = \mathfrak{p} \ltimes \mathfrak{n}_-$  and invariants of  $(\mathfrak{g}_e, \text{ad}^*)$ , where  $e \in \mathfrak{g}$  is Richardson with polarisation  $\mathfrak{p}$ .

Let  $e \in \mathfrak{g}$  be a nonzero nilpotent element and  $\{e, h, f\}$  be an  $\mathfrak{sl}_2$ -triple in  $\mathfrak{g}$  (i.e.,  $[e, f] = h$ ,  $[h, e] = 2e$ ,  $[h, f] = -2f$ ). Then  $\kappa(e, f) = 1$ . If  $V \subset \mathfrak{g}$  is the orthogonal complement of  $e$  with respect to  $\kappa$ , then  $\mathfrak{g} = \mathbb{K}f \oplus V$  and  $\mathfrak{g}_e \subset V$ .

**Lemma 3.4** ([18, Lemma A.1]). *Let  $\mathcal{H} \in \mathcal{S}(\mathfrak{g})^G$  be homogeneous, of degree  $m$ . Consider the decomposition  $\mathcal{H} = \sum_{i \in \mathbb{N}} f^{m-i} \mathcal{H}_i$ , where  $\mathcal{H}_i \in \mathcal{S}^i(V)$ . If  $f^{m-k} \mathcal{H}_k$  is the nonzero summand with minimal  $k$ , then  $\mathcal{H}_k \in \mathcal{S}^k(\mathfrak{g}_e) \subset \mathcal{S}^k(V)$ . Moreover, this  $\mathcal{H}_k$  is  $G_e$ -invariant.*

*Remark 3.5.* The polynomial  $\mathcal{H}_k$  coincides with certain polynomial  ${}^e\mathcal{H}$  that can be constructed via the Slodowy slice associated with  $\{e, h, f\}$ , see [18, Prop. 0.1 & Cor. A.2]. Therefore, we use notation  ${}^e\mathcal{H}$  for the above polynomial  $\mathcal{H}_k$  associated with  $\mathcal{H}$ .

From now on, we assume that  $e \in \mathfrak{g}$  is Richardson and  $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$  is a polarisation of  $e$ . To any homogeneous  $\mathcal{H} \in \mathcal{S}(\mathfrak{g})^G$ , we have attached two polynomials,  $\mathcal{H}^\bullet \in \mathcal{S}(\mathfrak{q})^Q$  (see Section 1) and  ${}^e\mathcal{H} \in \mathcal{S}(\mathfrak{g}_e)^{G_e}$ . To provide a link between them, one has to adjust the above construction of  ${}^e\mathcal{H}$  to the decompositions  $\mathfrak{q} = \mathfrak{p} \oplus \mathfrak{n}_-$  and  $\mathfrak{q}^* = \mathfrak{n} \oplus \mathfrak{p}_-$ . If  $f \in \mathfrak{n}_-$ , then no special adjustment is needed. Otherwise, we take  $y \in \mathfrak{n}_-$  such that  $\kappa(y, e) = 1$ . Then  $f - y \in V$  and  $\mathfrak{g} = \mathbb{K}y \oplus V$ . If  $\mathcal{H} = \sum_i y^{m-i} \tilde{\mathcal{H}}_i$  with  $\tilde{\mathcal{H}}_i \in \mathcal{S}^i(V)$ , then it is easily seen that the nonzero summand with minimal  $i$  occurs for  $i = k$  and  $\tilde{\mathcal{H}}_k = \mathcal{H}_k$ . That is, we can (and will) use such an  $y \in \mathfrak{n}_-$  in place of  $f$  for defining  ${}^e\mathcal{H}$ .

Recall that  $\mathfrak{q}^* = \mathfrak{n} \oplus \mathfrak{p}_-$  is a sum of  $P$ -modules and the second summand here is also a  $Q$ -submodule. Since  $e \in \mathfrak{n}$ ,  $e + \mathfrak{p}_-$  is an affine subspace of  $\mathfrak{q}^*$  and we identify  $\mathbb{K}[e + \mathfrak{p}_-]$  with  $\mathbb{K}[\mathfrak{p}_-] = \mathcal{S}(\mathfrak{p})$ . Recall also that  $\mathfrak{g}_e = \mathfrak{p}_e \subset \mathfrak{p}$  and therefore  $\mathcal{S}(\mathfrak{g}_e) \subset \mathcal{S}(\mathfrak{p})$ . Hence  ${}^e\mathcal{H}$  can be thought of as element of  $\mathcal{S}(\mathfrak{p})$  that belongs to the subalgebra  $\mathcal{S}(\mathfrak{g}_e)^{G_e}$ .

**Proposition 3.6.** *If  $\mathcal{H} \in \mathcal{S}(\mathfrak{g})^G$  is homogeneous, then under the above identifications, we have  $\mathcal{H}^\bullet|_{e+\mathfrak{p}_-} = {}^e\mathcal{H}$ . In particular,  $\mathcal{H}^\bullet|_{e+\mathfrak{p}_-}$  belongs to the subalgebra  $\mathcal{S}(\mathfrak{g}_e)^{G_e}$  of  $\mathcal{S}(\mathfrak{p})$  and  $\deg_{\mathfrak{p}}(\mathcal{H}^\bullet) = \deg({}^e\mathcal{H})$ .*

*Proof.* As in Lemma 3.4, assume that  $\deg(\mathcal{H}) = m$  and  $\deg({}^e\mathcal{H}) = k$ . By definition,  $\mathcal{H}^\bullet$  is the nonzero bi-homogeneous component of  $\mathcal{H}$  with highest  $\mathfrak{n}_-$ -degree. By Theorem 1.1,  $\mathcal{H}^\bullet$  is  $Q$ -invariant and hence  $P$ -invariant. Since  $\overline{P \cdot e} = \mathfrak{n}$ , we have  $\overline{P(e + \mathfrak{p}_-)} = \mathfrak{q}^*$ , and  $\mathcal{H}^\bullet$  does not vanish on  $e + \mathfrak{p}_-$ . In view of the inclusion  $\mathfrak{g}_e \subset \mathfrak{p}$ , Lemma 3.4 (with  $f$  replaced by  $y$ ) shows that  $\mathcal{H}$  has a non-zero summand of bi-degree  $(k, m - k)$  with respect to  $(\mathfrak{p}, \mathfrak{n}_-)$ . Hence  $\deg_{\mathfrak{n}_-}(\mathcal{H}) \geq m - k$ . On the other hand,  $\mathfrak{p} \subset V$  and  $\mathcal{H}^\bullet|_{e + \mathfrak{p}_-}$  has degree at least  $k$  as element of  $\mathcal{S}(\mathfrak{p})$ . Therefore  $\deg_{\mathfrak{n}_-} \mathcal{H} = m - k$  and  $\mathcal{H}^\bullet|_{e + \mathfrak{p}_-} = {}^e\mathcal{H}$ .  $\square$

Since  $\mathfrak{p}_-$  is a  $Q$ -submodule of  $\mathfrak{q}^*$ , the affine subspace  $e + \mathfrak{p}_-$  is invariant with respect to the subgroup  $P_e \ltimes N_-^a \subset Q$ . Therefore, one has a well-defined homomorphism

$$\psi : \mathcal{S}(\mathfrak{q})^Q \rightarrow \mathbb{k}[e + \mathfrak{p}_-]^{P_e \ltimes N_-^a}$$

that takes  $\mathcal{H}$  to  $\mathcal{H}|_{e + \mathfrak{p}_-}$ . Because  $\overline{Q \cdot (e + \mathfrak{p}_-)} = \mathfrak{q}^*$ ,  $\psi$  is injective. It is also clear that, under the identification  $\mathbb{k}[e + \mathfrak{p}_-] \simeq \mathcal{S}(\mathfrak{p})$ , the image of  $\psi$  belong to  $\mathcal{S}(\mathfrak{p})^{P_e}$ . Hence  $\psi$  can be thought of as a homomorphism from  $\mathcal{S}(\mathfrak{q})^Q$  to  $\mathcal{S}(\mathfrak{p})^{P_e}$ .

**Theorem 3.7.** *Let  $\mathcal{F}_1, \dots, \mathcal{F}_l \in \mathcal{S}(\mathfrak{g})^G$  be the basic invariants and  $\mathfrak{q}$  the parabolic contraction of  $\mathfrak{g}$  defined by  $\mathfrak{p}$ . Then*

$$(3.1) \quad \mathcal{F}_1^\bullet, \dots, \mathcal{F}_l^\bullet \text{ are algebraically independent if and only if } {}^e\mathcal{F}_1, \dots, {}^e\mathcal{F}_l \text{ are.}$$

*If the equivalent conditions of (3.1) hold, then*

- (i)  $\mathcal{S}(\mathfrak{q})^Q \supset \mathbb{k}[\mathcal{F}_1^\bullet, \dots, \mathcal{F}_l^\bullet]$  is an algebraic extension;
- (ii)  $\psi(\mathcal{S}(\mathfrak{q})^Q) \subset \mathcal{S}(\mathfrak{g}_e)^{P_e}$ ;
- (iii) Moreover, if  $\mathcal{S}(\mathfrak{g}_e)^{G_e} = \mathcal{S}(\mathfrak{g}_e)^{P_e}$  and this algebra is freely generated by  ${}^e\mathcal{F}_1, \dots, {}^e\mathcal{F}_l$ , then  $\mathcal{S}(\mathfrak{q})^Q$  is freely generated by  $\mathcal{F}_1^\bullet, \dots, \mathcal{F}_l^\bullet$ .

*Proof.* The equivalence of two conditions in (3.1) follows from the fact that  $\psi$  is injective and  $\psi(\mathcal{F}_i^\bullet) = {}^e\mathcal{F}_i$  (see Proposition 3.6).

(i) Since  $\text{ind } \mathfrak{q} = l$  (see Theorem 3.1), one always has  $\text{tr.deg } \mathcal{S}(\mathfrak{q})^Q \leq l$ . Therefore  $\text{tr.deg } \mathcal{S}(\mathfrak{q})^Q = l$ , and the assertion follows.

(ii) It follows from (i) that  $\psi(\mathcal{S}(\mathfrak{q})^Q) \supset \mathbb{k}[{}^e\mathcal{F}_1, \dots, {}^e\mathcal{F}_l]$  is an algebraic extension. Because  ${}^e\mathcal{F}_i \in \mathcal{S}(\mathfrak{g}_e)^{G_e}$  and  $\mathcal{S}(\mathfrak{g}_e)$  is algebraically closed in  $\mathcal{S}(\mathfrak{p})$ , we have

$$(3.2) \quad \psi(\mathcal{S}(\mathfrak{q})^Q) \subset \mathcal{S}(\mathfrak{g}_e) \cap \mathcal{S}(\mathfrak{p})^{P_e} = \mathcal{S}(\mathfrak{g}_e)^{P_e}.$$

(iii) Here we have  $\mathcal{S}(\mathfrak{g}_e)^{G_e} = \mathbb{k}[{}^e\mathcal{F}_1, \dots, {}^e\mathcal{F}_l] \subset \psi(\mathcal{S}(\mathfrak{q})^Q) \subset \mathcal{S}(\mathfrak{g}_e)^{P_e} = \mathcal{S}(\mathfrak{g}_e)^{G_e}$ .

Whence  $\psi(\mathcal{S}(\mathfrak{q})^Q) = \mathbb{k}[{}^e\mathcal{F}_1, \dots, {}^e\mathcal{F}_l]$  and therefore  $\mathcal{S}(\mathfrak{q})^Q = \mathbb{k}[\mathcal{F}_1^\bullet, \dots, \mathcal{F}_l^\bullet]$ .  $\square$

**Remark 3.8.** In view of the above theorem, it is important to know when the  $P_e$ - and  $G_e$ -invariants in  $\mathcal{S}(\mathfrak{g}_e)$  coincide. This condition is weaker than the coincidence of  $P_e$  and  $G_e$ . For any Richardson element  $e \in \mathfrak{n} = \mathfrak{p}^{nil}$ , one can consider the chain of groups

$$G_e^o \subset P_e \subset G_e,$$

where  $G_e^o$  is the identity component of  $G_e$ , and the corresponding chain of rings of invariants

$$\mathcal{S}(\mathfrak{g}_e)^{G_e} \subset \mathcal{S}(\mathfrak{g}_e)^{P_e} \subset \mathcal{S}(\mathfrak{g}_e)^{G_e^o} =: \mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}.$$

All these inclusions can be strict. As is well known, the equality  $P_e = G_e$  has the following geometric meaning. The cotangent bundle  $T^*(G/P) \simeq G \times_P \mathfrak{n}$  has the natural collapsing

$$\phi : G \times_P \mathfrak{n} \rightarrow G \cdot \mathfrak{n} = \overline{G \cdot e} \subset \mathfrak{g}$$

such that the fibre  $\phi^{-1}(e)$  has cardinality  $\#(G_e/P_e)$ , see e.g. [1, § 7]. Therefore,  $\phi$  is birational (and thereby is a resolution of singularities of  $\overline{G \cdot e}$ ) if and only if  $G_e = P_e$ . It is also known that if  $e$  is even (i.e., the weighted Dynkin diagram of  $e$  has only even labels), then  $P_e = G_e$  for the Dynkin-Jacobson-Morozov parabolic subalgebra associated with  $e$ .

#### 4. PARABOLIC CONTRACTIONS FOR CLASSICAL LIE ALGEBRAS

In this section, we prove that (i) if  $\mathfrak{g}$  is a simple Lie algebra of type  $\mathbf{A}_l$  or  $\mathbf{C}_l$ , then  $\mathcal{S}(\mathfrak{q})^Q$  is a polynomial algebra for any  $\mathfrak{p}$ ; (ii) if  $\mathfrak{g}$  is of type  $\mathbf{B}_l$ , then  $\mathcal{S}(\mathfrak{q})^Q$  is a polynomial algebra whenever the Levi subalgebra of  $\mathfrak{p}$  is of the form  $\mathfrak{gl}_{n_1} \oplus \dots \oplus \mathfrak{gl}_{n_t}$ , where  $n_1, \dots, n_t$  are odd.

It is quite common in invariant-theoretic problems that a certain method works well in types  $\mathbf{A}_l$  and  $\mathbf{C}_l$  and does not extend in full generality to other simple Lie algebras. This has happened in [18] in connection with the study of symmetric invariants of centralisers, and also evinces here, because our approach relies on results of that paper. The same phenomenon also manifests itself in [9, 10], where an explicit description of a Borel (or parabolic) contraction of simple  $SL_{l+1}$  or  $Sp_{2l}$ -modules to “highest weight”  $Q$ -modules is obtained. No similar results are known so far in other types.

**Theorem 4.1.** *Suppose that  $\mathfrak{g}$  is either  $\mathfrak{sl}_{l+1}$  or  $\mathfrak{sp}_{2l}$  and  $\mathfrak{q}$  is a parabolic contraction of  $\mathfrak{g}$ . Then there exist basic invariants  $\mathcal{F}_1, \dots, \mathcal{F}_l \in \mathcal{S}(\mathfrak{g})^G$  such that  $\mathcal{F}_1^\bullet, \dots, \mathcal{F}_l^\bullet$  freely generate  $\mathcal{S}(\mathfrak{q})^Q$  and satisfy the Kostant equality in  $\mathfrak{q}$ , and the equality  $\sum_{i=1}^l \deg_{\mathfrak{n}_-}(\mathcal{F}_i^\bullet) = \dim \mathfrak{n}$  holds.*

*Proof.* For  $\mathfrak{g} = \mathfrak{sl}_{l+1}$  or  $\mathfrak{sp}_{2l}$ , let  $\mathcal{F}_1, \dots, \mathcal{F}_l \in \mathcal{S}(\mathfrak{g})^G$  be the coefficients of the characteristic polynomial of a matrix in  $\mathfrak{g}$ . It is proved in [18, Theorems 4.2 & 4.4] that  ${}^e\mathcal{F}_1, \dots, {}^e\mathcal{F}_l$  are algebraically independent for every nonzero nilpotent element  $e \in \mathfrak{g}$  and

$$(4.1) \quad \mathcal{S}(\mathfrak{g}_e)^{G_e} = \mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e} = \mathbb{K}[{}^e\mathcal{F}_1, \dots, {}^e\mathcal{F}_l].$$

This also shows that if  $e \in \mathfrak{n}$  is Richardson, then  $\mathcal{S}(\mathfrak{g}_e)^{G_e} = \mathcal{S}(\mathfrak{g}_e)^{P_e}$ . Therefore, applying Theorem 3.7 to our Richardson element  $e \in \mathfrak{n}$ , we conclude that, for the above-mentioned choice of basic invariants,  $\mathcal{S}(\mathfrak{q})^Q$  is freely generated by  $\mathcal{F}_1^\bullet, \dots, \mathcal{F}_l^\bullet$ .

Since  $\mathcal{F}_1^\bullet, \dots, \mathcal{F}_l^\bullet$  are algebraically independent, it follows from Theorem 3.3 that  $\mathcal{F}_1^\bullet, \dots, \mathcal{F}_l^\bullet$  satisfy the Kostant equality and  $\sum_{i=1}^l \deg_{\mathfrak{n}_-}(\mathcal{F}_i^\bullet) = \dim \mathfrak{n}$ .  $\square$

To describe our results in the orthogonal case, we introduce some terminology. We say that a parabolic subalgebra of  $\mathfrak{g} = \mathfrak{so}_{2l+1}$  is *admissible*, if the Levi subalgebras of  $\mathfrak{p}$  are of the form  $\mathfrak{gl}_{n_1} \oplus \dots \oplus \mathfrak{gl}_{n_t}$ , where  $n_1, \dots, n_t$  are odd. The corresponding parabolic contractions and Richardson orbits are said to be *admissible*, too.

**Theorem 4.2.** *Let  $\mathfrak{q}$  be an admissible parabolic contraction of  $\mathfrak{so}_{2l+1}$ . Then there exist basic invariants  $\mathcal{F}_1, \dots, \mathcal{F}_l \in \mathcal{S}(\mathfrak{g})^G$  such that  $\mathcal{F}_1^\bullet, \dots, \mathcal{F}_l^\bullet$  freely generate  $\mathcal{S}(\mathfrak{q})^Q$  and satisfy the Kostant equality in  $\mathfrak{q}$ , and the equality  $\sum_{i=1}^l \deg_{\mathfrak{n}}(\mathcal{F}_i^\bullet) = \dim \mathfrak{n}$  holds.*

*Proof.* As in Theorem 4.1, take  $\mathcal{F}_1, \dots, \mathcal{F}_l$  to be the coefficients of the characteristic polynomial of a matrix in  $\mathfrak{g}$ . Suppose that  $e \in \mathfrak{g}$  is given by the partition  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_t)$  of  $2l + 1$  such that  $\lambda_1$  is odd and all other parts are even. (Recall that, for the nilpotent elements of  $\mathfrak{so}(V)$ , each even part of  $\lambda$  occurs an even number of times.) By [18, Theorem 4.7],  $\mathcal{F}_1, \dots, \mathcal{F}_l$  is a “very good generating system” for  $e$ , which, in view of [18, Theorem 2.2], implies that  ${}^e\mathcal{F}_1, \dots, {}^e\mathcal{F}_l$  are algebraically independent and (4.1) holds.

Using [12, 4.2], one verifies that the above elements  $e$  are exactly the admissible Richardson elements. In this case,  $\hat{\lambda}$  is of the form  $(\mu_1^2, \mu_2^2, \dots, \mu_s^2, 1^{2k+1})$ , where all  $\mu_i$  are odd and  $\mu_1 \geq \mu_2 \geq \dots \geq \mu_s \geq 3$ , and then  $\mathfrak{l} = \mathfrak{gl}_{\mu_1} \oplus \dots \oplus \mathfrak{gl}_{\mu_t} \oplus (\mathfrak{gl}_1)^k$ . Therefore, Theorem 3.7 can be applied to the admissible parabolic subalgebras  $\mathfrak{p}$  and parabolic contractions  $\mathfrak{q}$ , and we conclude that  $\mathcal{S}(\mathfrak{q})^Q$  is freely generated by  $\mathcal{F}_1^\bullet, \dots, \mathcal{F}_l^\bullet$ . The rest is the same as in Theorem 4.1.  $\square$

**Remark.** Theorem 4.7 in [18], which is used in the previous proof, refers also to similar nilpotent elements in  $\mathfrak{so}_{2l}$ . But all those elements are not Richardson.

For the parabolic contractions described in Theorems 4.1 and 4.2, the basic invariants in  $\mathcal{S}(\mathfrak{q})^Q$  have the same degrees as the basic invariants in  $\mathcal{S}(\mathfrak{g})^G$ . But the algebra  $\mathcal{S}(\mathfrak{q})^Q$  is bi-graded, and our next goal is to determine the bi-degrees of  $\mathcal{F}_1^\bullet, \dots, \mathcal{F}_l^\bullet$  with respect to decomposition (1.1) in the corresponding cases. By Proposition 3.6, we have  $\deg_{\mathfrak{p}}(\mathcal{F}_i^\bullet) = \deg({}^e\mathcal{F}_i)$ . For all (resp. some) nilpotent elements in  $\mathfrak{sl}_{l+1}$  or  $\mathfrak{sp}_{2l}$  (resp.  $\mathfrak{so}_{2l+1}$ ), there is an explicit algorithm for computing the degrees of  ${}^e\mathcal{F}_1, \dots, {}^e\mathcal{F}_l$  [18, Sect. 4]. We prove below that, for the (admissible) Richardson elements, this can be restated in terms of a Levi subalgebra  $\mathfrak{l} \subset \mathfrak{p}$ .

**Proposition 4.3.** *Let  $e \in \mathfrak{g}$  be a Richardson element with a polarisation  $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ , where*

- $\mathfrak{p}$  is any parabolic subalgebra, if  $\mathfrak{g} = \mathfrak{sl}_{l+1}$  or  $\mathfrak{sp}_{2l}$ ;
- $\mathfrak{p}$  is admissible, if  $\mathfrak{g} = \mathfrak{so}_{2l+1}$ .

*Then the multiset of degrees of  ${}^e\mathcal{F}_1, \dots, {}^e\mathcal{F}_l$  is the same as the multiset of degrees of the basic  $L$ -invariants in  $\mathcal{S}(\mathfrak{l})$ .*

*Proof.* 1) To simplify exposition, we work here with  $\mathfrak{gl}_{l+1}$  in place of  $\mathfrak{sl}_{l+1}$ . Then  $\{\deg \mathcal{F}_j\} = \{1, 2, \dots, l+1\}$ . Recall that all nilpotent elements of  $\mathfrak{gl}_{l+1}$  are Richardson. Let  $e \in \mathfrak{gl}_{l+1}$  correspond to the partition  $\lambda = (\lambda_1, \dots, \lambda_t)$ , where  $\sum_i \lambda_i = l+1$  and  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_t > 0$ . Then

$$\#\{j \mid \deg({}^e\mathcal{F}_j) = i\} = \lambda_i,$$

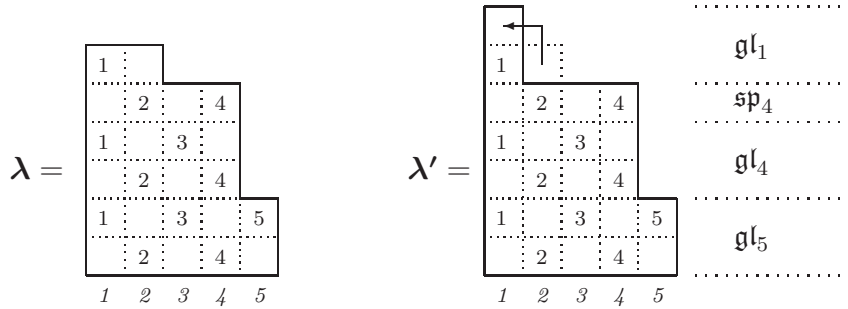
see [18, p. 368], i.e., the multiset of degrees of the  ${}^e\mathcal{F}_j$ 's is  $\{1^{\lambda_1}, 2^{\lambda_2}, \dots, t^{\lambda_t}\}$ . More precisely, if  $\lambda_1 + \dots + \lambda_{i-1} + 1 \leq \deg(\mathcal{F}_j) \leq \lambda_1 + \dots + \lambda_i$ , then  $\deg({}^e\mathcal{F}_j) = i$ . On the other hand, if  $\hat{\lambda} = (\hat{\lambda}_1, \dots, \hat{\lambda}_s)$  is the dual partition, then  $\mathfrak{l} \simeq \mathfrak{gl}_{\hat{\lambda}_1} \oplus \dots \oplus \mathfrak{gl}_{\hat{\lambda}_s}$ . Therefore, the basic invariants of degree  $i$  in  $\mathcal{S}(\mathfrak{l})^L$  occur with multiplicity  $\#\{j \mid \hat{\lambda}_j \geq i\} = \lambda_i$ .

2) If  $\mathfrak{g} = \mathfrak{sp}_{2l}$ , then  $\{\deg \mathcal{F}_j\} = \{2, 4, \dots, 2l\}$  and there is a similar algorithm to determine  $\deg({}^e\mathcal{F}_j)$  for all  $e$  [18, 4.3]. Let  $\lambda = (\lambda_1, \dots, \lambda_t)$  be the partition of  $2l$  corresponding to  $e$ . (Recall that, for the nilpotent elements of  $\mathfrak{sp}(V)$ , each odd part occurs an even number of times.) Then we have  $\deg({}^e\mathcal{F}_j) = i$ , if  $\lambda_1 + \dots + \lambda_{i-1} + 1 \leq \deg(\mathcal{F}_j) \leq \lambda_1 + \dots + \lambda_i$ .

By [12, 4.1],  $e$  is Richardson if and only if  $\lambda$  satisfies the following conditions:

- (1) either all  $\lambda_i$  are even, or  $r := \max\{j \mid \lambda_j \text{ is odd}\}$  is even  
(set  $r = 0$  if all parts are even);
- (2)  $\lambda_{2j-1}, \lambda_{2j}$  have the same parity for  $2j \leq r$ ;
- (3) if  $\lambda_{2j}, \lambda_{2j+1}$  are even (for  $2j < r$ ), then  $\lambda_{2j} \geq \lambda_{2j+1} + 2$ .

For the Richardson elements, the above algorithm for finding  $\deg({}^e\mathcal{F}_j)$  can graphically be presented via the chessboard filling of the Young diagram of  $\lambda$ . See the left figure below, where  $\lambda = (6, 6, 5, 5, 2)$  and the parts  $\lambda_i$  represent the columns of the diagram. For this diagram, one obtains  $\{\deg({}^e\mathcal{F}_j)\} = \{1^3, 2^3, 3^2, 4^3, 5\}$ .



To describe a Levi subalgebra  $\mathfrak{l}$  corresponding to such a  $\lambda$ , one proceeds as follows. Take all even pairs  $\lambda_{2j-1}, \lambda_{2j}$  ( $2j \leq r$ ) and replace them with  $\lambda_{2j-1} + 1, \lambda_{2j} - 1$ . Because of the conditions, one obtains a partition  $\lambda'$  of the form  $\lambda' = (\underbrace{\lambda'_1, \dots, \lambda'_r}_{\text{odd}}, \underbrace{\lambda_{r+1}, \dots, \lambda_t}_{\text{even}})$ . Note that

$\lambda'_r = \lambda_r$ . The dual partition  $\hat{\lambda}' = \{\hat{\lambda}_1, \dots, \hat{\lambda}_s\}$  determines one of the Levi subalgebras corresponding to  $e$ . Namely,  $\#\{j \mid \hat{\lambda}_j = i\}$  is even unless  $i = r$ , and each pair of parts of equal size  $\hat{\lambda}_j$  gives rise to the summand  $\mathfrak{gl}_{\hat{\lambda}_j}$  in  $\mathfrak{l}$ . The only non-paired part of size  $r$  gives rise to the summand  $\mathfrak{sp}_r$  in  $\mathfrak{l}$ . We may think of parts of  $\hat{\lambda}'$  as the rows of  $\lambda'$ . Then

the consecutive pairs of equal rows below or above the level  $\lambda_r$  represent the summands of the form  $\mathfrak{gl}_{\hat{\lambda}_j}$ , and our graphical algorithm shows that the corresponding pair of rows contain boxes filled with numbers  $1, 2, \dots, \hat{\lambda}_j$ ; while the remaining row of length  $r$  at level  $\lambda_r$  contains numbers  $2, 4, \dots, r$ . It is important that the passage from  $\lambda$  to  $\lambda'$  consists in moving only empty boxes! (See the right figure above, where  $r = 4$  and  $\mathfrak{l}$  is equal to  $\mathfrak{gl}_5 \oplus \mathfrak{gl}_4 \oplus \mathfrak{gl}_1 \oplus \mathfrak{sp}_4$ .) This shows that the assertion holds for this specific Levi subalgebra associated with  $e$ . By [12], all other Levi subalgebras (if any) are obtained by the following alterations: If  $\mathfrak{l}$  contains the summands  $\mathfrak{sp}_r \oplus \mathfrak{gl}_{r+2}$ , then they can be replaced with  $\mathfrak{sp}_{r+2} \oplus \mathfrak{gl}_{r+1}$  (all other summands remain intact). Clearly, this step does not change the degrees of basic invariants in  $\mathcal{S}(\mathfrak{l})^L$ .

3) If  $\mathfrak{g} = \mathfrak{so}_{2l+1}$ , then  $\{\deg \mathcal{F}_j\} = \{2, 4, \dots, 2l\}$ . For the admissible Richardson elements  $e$ , the algorithm for computing  $\deg({}^e\mathcal{F}_j)$  is the same as in part 2), see [18, 4.4]. If  $\lambda = (\lambda_1, \dots, \lambda_t)$  is admissible, i.e.,  $\lambda_1$  is odd and all other parts are even, then  $\lambda_2 = \lambda_3, \lambda_4 = \lambda_5$ , etc., and we obtain

$$\#\{j \mid \deg({}^e\mathcal{F}_j) = i\} = \begin{cases} \lfloor \lambda_1/2 \rfloor, & i = 1 \\ \lambda_i/2, & i > 1. \end{cases}$$

Then  $\hat{\lambda} = (\hat{\lambda}_1, \dots, \hat{\lambda}_{2s+2k+1})$ , where  $\lambda_1 = 2s + 2k + 1$ ,  $\lambda_2 = 2s$ ,  $\hat{\lambda}_{2i-1} = \hat{\lambda}_{2i}$  for  $i = 1, \dots, s$  and  $\hat{\lambda}_{2s+1} = \dots = \hat{\lambda}_{2k+2s+1} = 1$ . In this case,  $\mathfrak{l} = \mathfrak{gl}_{\hat{\lambda}_2} \oplus \mathfrak{gl}_{\hat{\lambda}_4} \oplus \dots \oplus \mathfrak{gl}_{\hat{\lambda}_{2s}} \oplus (\mathfrak{gl}_1)^k$ . Therefore, the basic invariants of degree  $i$  in  $\mathcal{S}(\mathfrak{l})^L$  occur with multiplicity

$$\begin{cases} s + k, & i = 1 \\ \#\{j \mid \hat{\lambda}_{2j} \geq i\}, & i > 1. \end{cases}$$

It remains to observe that  $s + k = \lfloor \lambda_1/2 \rfloor$  and, for  $i > 1$ , we have  $\#\{j \mid \hat{\lambda}_{2j} \geq i\} = \frac{1}{2}\#\{j \mid \hat{\lambda}_j \geq i\} = \frac{1}{2}\lambda_i$ .  $\square$

By Theorems 4.1 and 4.2, the sum of  $\deg_{\mathfrak{n}_-}(\mathcal{F}_i^\bullet)$  equals  $\dim \mathfrak{n}$ . However, Proposition 4.3 provides another approach to this equality.

**Corollary 4.4.** *For the bi-homogeneous basic invariants  $\mathcal{F}_1^\bullet, \dots, \mathcal{F}_l^\bullet$ , we have  $\sum_i \deg_{\mathfrak{n}_-}(\mathcal{F}_i^\bullet) = \dim \mathfrak{n}$  and  $\sum_i \deg_{\mathfrak{p}}(\mathcal{F}_i^\bullet) = \dim \mathfrak{b}(\mathfrak{l})$ , where  $\mathfrak{b}(\mathfrak{l})$  is a Borel subalgebra of  $\mathfrak{l}$ .*

*Proof.* Since  $\deg_{\mathfrak{p}}(\mathcal{F}_i^\bullet) = \deg({}^e\mathcal{F}_i)$ , the second equality follows immediately from the proposition. The rest follows from the equalities

$$\deg(\mathcal{F}_i) = \deg(\mathcal{F}_i^\bullet), \quad \sum_{i=1}^l \deg(\mathcal{F}_i) = \dim \mathfrak{b}, \quad \text{and} \quad \dim \mathfrak{b} = \dim \mathfrak{b}(\mathfrak{l}) + \dim \mathfrak{n}_-. \quad \square$$

**Example 4.5.** 1)  $\lambda = (6, 4, 2)$  determines a Richardson element in  $\mathfrak{sp}_{12}$ . Here  $\{\deg(\mathcal{F}_i)\} = \{2, 4, 6, 8, 10, 12\}$  and the algorithm transforms these degrees in  $\{\deg({}^e\mathcal{F}_i)\} = \{1, 1, 1, 2, 2, 3\}$ . This is in accordance with the fact that the corresponding Levi subalgebra is  $\mathfrak{gl}_3 \oplus \mathfrak{gl}_2 \oplus \mathfrak{gl}_1$ . Thus, the bi-degrees  $(\deg_{\mathfrak{p}} \mathcal{F}_i^\bullet, \deg_{\mathfrak{n}_-} \mathcal{F}_i^\bullet)$  of  $\{\mathcal{F}_i^\bullet\}$  are:

$$(1, 1), (1, 3), (1, 5), (2, 6), (2, 8), (3, 9).$$



2)  $\lambda = (3, 3, 1, 1)$  determines a Richardson element in  $\mathfrak{sp}_8$ . Here  $\{\deg(\mathcal{F}_i)\} = \{2, 4, 6, 8\}$  and the algorithm transforms these degrees in  $\{\deg({}^e\mathcal{F}_i)\} = \{1, 2, 2, 4\}$ . Accordingly, the corresponding Levi subalgebra is  $\mathfrak{sp}_4 \oplus \mathfrak{gl}_2$ . Thus, the bi-degrees  $(\deg_{\mathfrak{p}} \mathcal{F}_i^\bullet, \deg_{\mathfrak{n}_-} \mathcal{F}_i^\bullet)$  of  $\{\mathcal{F}_i^\bullet\}$  are:  $(1, 1), (2, 2), (2, 4), (4, 4)$ .

3)  $\lambda = (5, 4, 4, 2, 2)$  determines an admissible Richardson element in  $\mathfrak{so}_{17}$ . Here  $\{\deg(\mathcal{F}_i)\} = \{2, 4, 6, 8, 10, 12, 14, 16\}$  and the algorithm transforms these numbers in  $\{\deg({}^e\mathcal{F}_i)\} = \{1^2, 2^2, 3^2, 4, 5\}$ . This corresponds to the fact that  $\mathfrak{l} = \mathfrak{gl}_5 \oplus \mathfrak{gl}_3$ .

*Remark 4.6.* The reason for our partial success is that there is a general relationship between  $\mathcal{H}^\bullet$  and  $\mathcal{H}$  (Prop. 3.6) and the polynomials  ${}^e\mathcal{F}_1, \dots, {}^e\mathcal{F}_l$  are algebraically independent for all (resp. admissible) Richardson elements  $e$  in  $\mathfrak{sl}_{l+1}$  and  $\mathfrak{sp}_{2l}$  (resp.  $\mathfrak{so}_{2l+1}$ ). However, for  $\mathfrak{g} = \mathfrak{so}_{2l}$ , there are Richardson elements  $e$  such that  ${}^e\mathcal{F}_1, \dots, {}^e\mathcal{F}_l$  are algebraically dependent for any choice of basic invariants  $\mathcal{F}_i$ . Namely, this happens for  $e \in \mathfrak{so}_{12}$  corresponding to the partition  $(5, 3, 2, 2)$ , see [18, Example 4.1]. (Here  $\dim \mathfrak{g}_e = 18$  and the semisimple part of  $\mathfrak{l}$  is of type  $A_3$ .) For the corresponding parabolic contraction  $\mathfrak{q}$ ,  $\mathcal{F}_1^\bullet, \dots, \mathcal{F}_l^\bullet$  are also algebraically dependent, see (3.1). One can prove that  $\mathcal{S}(\mathfrak{q})^Q$  always has the transcendence degree  $l$ , hence here  $\mathcal{S}(\mathfrak{q})^Q$  is not generated by  $\mathcal{F}_1^\bullet, \dots, \mathcal{F}_l^\bullet$ . However, this does not necessarily mean that here  $\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$  or  $\mathcal{S}(\mathfrak{q})^Q$  cannot be a polynomial algebra.

## 5. MINIMAL PARABOLIC SUBALGEBRAS AND SUBREGULAR CONTRACTIONS

In this section  $\mathfrak{g}$  is a simple Lie algebra. Fix a triangular decomposition  $\mathfrak{g} = \mathfrak{u}_- \oplus \mathfrak{t} \oplus \mathfrak{u}$ , where  $\mathfrak{t}$  is a Cartan subalgebra and  $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{u}$ . Then  $\Delta$  is the root system of  $(\mathfrak{g}, \mathfrak{t})$ ,  $\Delta^+$  is the set of roots of  $\mathfrak{u}$ ,  $\Pi$  is the set of simple roots in  $\Delta^+$ , and  $\delta$  is the highest root in  $\Delta^+$ . Write  $\mathfrak{g}^\gamma$  for the root space corresponding to  $\gamma \in \Delta$ .

Let  $\mathfrak{p}$  be a minimal parabolic subalgebra of  $\mathfrak{g}$ , i.e.,  $\dim \mathfrak{p} = \dim \mathfrak{b} + 1$  and  $[\mathfrak{l}, \mathfrak{l}] \simeq \mathfrak{sl}_2$ . We assume that  $\mathfrak{p} = \mathfrak{b} \oplus \mathfrak{g}^{-\alpha}$  for some  $\alpha \in \Pi$ . Then  $\mathfrak{n} \oplus \mathfrak{g}^\alpha = \mathfrak{u}$ . If  $e \in \mathfrak{n}$  is Richardson, then  $\dim \mathfrak{g}_e = \dim \mathfrak{g} - 2 \dim \mathfrak{n} = l + 2$  and  $G \cdot e$  is the subregular nilpotent orbit. The parabolic contraction associated with  $\mathfrak{p}$  is said to be *subregular*, too. From now on,  $\mathfrak{q}$  is a subregular contraction of  $\mathfrak{g}$ . To exclude the case in which  $\mathfrak{p} = \mathfrak{g}$ , we assume below that  $l \geq 2$ .

Recall that the multiset  $\{\deg(\mathcal{F}_1), \dots, \deg(\mathcal{F}_l)\}$  does not depend on a particular choice of basic invariants in  $\mathcal{S}(\mathfrak{g})^G$ , and if  $\mathfrak{g}$  is simple, then there is a unique basic invariant of maximal degree. (This maximal degree equals the Coxeter number of  $\mathfrak{g}$ .) We assume below that  $\mathcal{F}_l$  has the maximal degree, so that  $\deg(\mathcal{F}_i) < \deg(\mathcal{F}_l)$  for  $i < l$ . The ordering of the previous basic invariants is irrelevant.

**Proposition 5.1.** *If  $\mathfrak{q}$  is a subregular contraction of  $\mathfrak{g}$ , then*

- (i)  $\deg_{\mathfrak{p}}(\mathcal{F}_i^\bullet) = 1$  for  $i = 1, \dots, l-1$  and  $\deg_{\mathfrak{p}}(\mathcal{F}_l^\bullet) = 2$ ,

(ii) the polynomials  $\mathcal{F}_1^\bullet, \dots, \mathcal{F}_l^\bullet$  are algebraically independent and satisfy the Kostant equality.

*Proof.* Recall that  $\mathcal{F}_i^\bullet$  is the bi-homogeneous component of  $\mathcal{F}_i$  with highest  $\mathfrak{n}_-$ -degree.

(i) Since  $P$  has a dense orbit in  $\mathfrak{n}$ , we have  $\mathcal{S}(\mathfrak{n}_-)^P = \mathbb{k}$ . Therefore the  $P$ -invariant  $\mathcal{F}_i^\bullet$  cannot belong to  $\mathcal{S}(\mathfrak{n}_-) \subset \mathcal{S}(\mathfrak{q})$  and hence  $\deg_{\mathfrak{n}_-}(\mathcal{F}_i^\bullet) \leq \deg(\mathcal{F}_i) - 1$  for all  $i$ .

Consider the bi-homogeneous component of  $\mathcal{F}_l$  with highest  $\mathfrak{u}_-$ -degree (with respect to the decomposition  $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{u}_-$ ), denoted by  $\mathcal{F}_l^\blacktriangle$ . It is known that  $\mathcal{F}_l^\blacktriangle = e_\delta \prod_{i=1}^l f_i^{a_i}$ , where  $e_\delta \in \mathfrak{g}^\delta$  is a highest root vector,  $f_i \in \mathfrak{g}^{-\alpha_i}$  for  $\alpha_i \in \Pi$ , and  $\delta = \sum_{i=1}^l a_i \alpha_i$ , see [19, Theorem 3.9 & Lemma 4.1]. That is,  $\mathcal{F}_l^\blacktriangle$  is a monomial and  $\deg_{\mathfrak{u}_-}(\mathcal{F}_l^\blacktriangle) = \deg(\mathcal{F}_l) - 1$ . Since  $\mathfrak{u}_- = \mathfrak{n}_- \oplus \mathfrak{g}^{-\alpha_i}$  for some  $i$  and all  $a_i$  are positive,  $\deg_{\mathfrak{n}_-}(\mathcal{F}_l^\blacktriangle) \leq \deg(\mathcal{F}_l) - 2$ . This also implies that  $\deg_{\mathfrak{n}_-}(\mathcal{F}_l^\bullet) \leq \deg(\mathcal{F}_l) - 2$ . Therefore,

$$(5.1) \quad \sum_{i=1}^l \deg_{\mathfrak{n}_-}(\mathcal{F}_i^\bullet) \leq \left( \sum_{i=1}^l \deg(\mathcal{F}_i) \right) - l - 1 = \dim \mathfrak{n}.$$

By Theorem 3.3, we have  $\sum \deg_{\mathfrak{n}_-}(\mathcal{F}_i^\bullet) \geq \dim \mathfrak{n}$ . Therefore one actually has the equality, which also means that  $\deg_{\mathfrak{n}_-}(\mathcal{F}_i^\bullet) = \deg(\mathcal{F}_i) - 1$  for  $i \leq l - 1$  and  $\deg_{\mathfrak{n}_-}(\mathcal{F}_l^\bullet) = \deg(\mathcal{F}_l) - 2$ .

(ii) By Theorem 3.3, the equality in (5.1) implies that  $\mathcal{F}_1^\bullet, \dots, \mathcal{F}_l^\bullet$  are algebraically independent and satisfy the Kostant equality.  $\square$

In the following lemma, we gather Lie-algebraic properties of the centraliser of a subregular nilpotent element.

**Lemma 5.2.** *Let  $e \in \mathfrak{g}$  be a subregular nilpotent element. Then*

- (i) *if  $\mathfrak{g}$  is not of type  $\mathbf{G}_2$ , then the centre of  $\mathfrak{g}_e$  is of dimension  $l - 1$ ; if  $\mathfrak{g}$  is of type  $\mathbf{G}_2$ , then the centre of  $\mathfrak{g}_e$  is two-dimensional (see [14, Theorem B]);*
- (ii) *if  $\mathfrak{g}$  is not of type  $\mathbf{G}_2$ , then  $\dim[\mathfrak{g}_e, \mathfrak{g}_e] > 1$ .*
- (iii) *if  $\mathfrak{g}$  is of type  $\mathbf{G}_2$ , then  $\mathfrak{g}_e$  is the direct sum of  $\mathbb{k}e$  and a three-dimensional Heisenberg Lie algebra  $\mathcal{H}_3$ .*

*Proof.* (ii) Since  $l = \text{ind } \mathfrak{g}_e < \dim \mathfrak{g}_e = l + 2$ ,  $\mathfrak{g}_e$  is not abelian, i.e.,  $[\mathfrak{g}_e, \mathfrak{g}_e] \neq 0$ . Assume that  $\dim[\mathfrak{g}_e, \mathfrak{g}_e] = 1$ . Write  $\mathfrak{g}_e = \mathfrak{z}(\mathfrak{g}_e) \oplus \mathfrak{c}$ , where  $\mathfrak{z}(\mathfrak{g}_e)$  is the centre, and  $\mathfrak{c}$  is a three-dimensional complement. Since  $[\mathfrak{g}_e, \mathfrak{g}_e] = [\mathfrak{c}, \mathfrak{c}]$  is one-dimensional, the space  $\mathfrak{c}$  must contain a non-trivial central element. A contradiction!

(iii) Let  $\Pi = \{\alpha, \beta\}$ , where  $\alpha$  is short. One can take  $e = e_\beta + e_{3\alpha+\beta}$ . Then  $\mathcal{H}_3 = \mathfrak{g}^{\alpha+\beta} \oplus \mathfrak{g}^{2\alpha+\beta} \oplus \mathfrak{g}^{3\alpha+2\beta}$ .  $\square$

**Proposition 5.3.** *Let  $P \subset G$  be a minimal parabolic subgroup and  $e \in \mathfrak{n}$  a subregular nilpotent element. Then  $\mathcal{S}(\mathfrak{g}_e)^{P_e} = \mathcal{S}(\mathfrak{g}_e)^{G_e}$  is freely generated by  ${}^e\mathcal{F}_1, \dots, {}^e\mathcal{F}_l$ .*

*Proof.* By Propositions 3.6 and 5.1(i), we have  $\deg({}^e\mathcal{F}_i) = 1$  for  $i \leq l-1$  and  $\deg({}^e\mathcal{F}_l) = 2$ . In particular,  ${}^e\mathcal{F}_1, \dots, {}^e\mathcal{F}_{l-1}$  are just elements of  $\mathfrak{g}_e$ . Moreover, Proposition 5.1(ii) implies that  ${}^e\mathcal{F}_1, \dots, {}^e\mathcal{F}_l$  are algebraically independent. (This also follows from the fact that  $\sum_{i=1}^l \deg({}^e\mathcal{F}_i) = l+1 = \frac{1}{2}(\dim \mathfrak{g}_e + \text{ind } \mathfrak{g}_e)$ , see [18, Theorem 2.1].)

Recall that all  ${}^e\mathcal{F}_i$  are  $G_e$ -invariant and hence  ${}^e\mathcal{F}_1, \dots, {}^e\mathcal{F}_{l-1}$  are linearly independent central elements of  $\mathfrak{g}_e$ . Then  $\mathfrak{z} := \text{span}\{{}^e\mathcal{F}_1, \dots, {}^e\mathcal{F}_{l-1}\}$  is a central subalgebra of  $\mathfrak{g}_e$ .

- Suppose that  $\mathfrak{g}$  is not of type  $\mathbf{G}_2$ . Then  $\mathfrak{z} = \mathfrak{z}(\mathfrak{g}_e)$  is the centre of  $\mathfrak{g}_e$ . Consider the coadjoint representation of  $G_e^o$  in  $\mathfrak{g}_e^*$ . Since  $\dim \mathfrak{g}_e = \text{ind } \mathfrak{g}_e + 2$ , the  $G_e^o$ -orbits in  $\mathfrak{g}_e^*$  are of dimension 2 and 0. Since  $G_e^o$  is connected, the union of 0-dimensional orbits is just the subspace  $V$  of  $\mathfrak{g}_e$ -fixed points, i.e.,  $V = \{\xi \in \mathfrak{g}_e^* \mid x \star \xi = 0 \ \forall x \in \mathfrak{g}_e\}$ . For a linear form  $\xi$ , one readily verifies that  $\xi \in V$  if and only if  $\xi$  vanishes on  $[\mathfrak{g}_e, \mathfrak{g}_e]$ . It then follows from Lemma 5.2(ii) that  $\text{codim } V \geq 2$ . In other words, the set of singular elements in  $\mathfrak{g}_e^*$  is of codimension  $\geq 2$ . Now, combining Theorems 2.1(iii) and 2.2 in [18], we obtain that

$$\mathcal{S}(\mathfrak{g}_e)^{G_e} = \mathcal{S}(\mathfrak{g}_e)^{G_e^o} = \mathbb{k}[{}^e\mathcal{F}_1, \dots, {}^e\mathcal{F}_l].$$

Since  $G_e^o \subset P_e \subset G_e$ , the assertion about  $P_e$ -invariants follows.

- Suppose that  $\mathfrak{g}$  is of type  $\mathbf{G}_2$ . Then  $\mathfrak{g}_e$  is the direct sum of  $\mathbb{k}e$  and a Heisenberg Lie algebra  $\mathcal{H}_3$ . Let  $(x, y, z)$  be a basis for  $\mathcal{H}_3$  such that  $[x, y] = z$  is the only non-trivial bracket. Then  $\mathfrak{z}(\mathfrak{g}_e) = \mathbb{k}e \oplus \mathbb{k}z$  and  $\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e} = \mathbb{k}[e, z]$ . Note that since  $\deg(\mathcal{F}_1) = 2$ , we have  ${}^e\mathcal{F}_1 = e$ . The component group  $G_e/G_e^o$  is the symmetric group  $\Sigma_3$  and it acts non-trivially on  $\mathfrak{z}(\mathfrak{g}_e)$ . This can be verified directly, using the element  $e$  indicated in the proof of Lemma 5.2(iii). Since  $e$  is a  $G_e$ -fixed vector, the line  $\mathbb{k}z \subset \mathfrak{z}(\mathfrak{g}_e)$  affords the unique non-trivial one-dimensional representation of  $\Sigma_3$ . Consequently,  $\mathcal{S}(\mathfrak{g}_e)^{G_e} = \mathbb{k}[e, z^2]$ , and because  ${}^e\mathcal{F}_1, {}^e\mathcal{F}_2$  are algebraically independent and  $\deg({}^e\mathcal{F}_2) = 2$ , we must have  ${}^e\mathcal{F}_2 = z^2 + ce^2$  for some  $c \in \mathbb{k}$ . Hence  $\mathcal{S}(\mathfrak{g}_e)^{G_e} = \mathbb{k}[{}^e\mathcal{F}_1, {}^e\mathcal{F}_2]$ .

There are two minimal parabolic subalgebras in  $\mathfrak{g}$  of type  $\mathbf{G}_2$ . For both of them,  $P_e$  is not connected and contains an element multiplying  $z$  by  $-1$ . This again can be verified via direct elementary calculations. (Cf. also Remark 5.4 below). Hence  $\mathcal{S}(\mathfrak{g}_e)^{P_e} = \mathbb{k}[e, z^2]$  in both cases, and we are done.  $\square$

*Remark 5.4.* For  $\alpha_i \in \Pi$ , let  $P_i$  denote the corresponding minimal parabolic in  $G$  and let  $e$  be a subregular element in  $\mathfrak{p}_i^{\text{nil}}$ . It was proved in [3, Prop. 4.2] that  $(P_i)_e = G_e$  if and only if  $\alpha_i$  is short (in the simply-laced case, all roots are assumed to be short). Moreover, using the explicit description of the Springer fibre of  $e$  as a *Dynkin curve* [24, p.147-148], one can show that if  $\alpha_i$  is long, then  $\#(G_e/(P_i)_e) = \|\alpha_i\|^2/\|\alpha_{\text{short}}\|^2$ . In the  $\mathbf{G}_2$ -case, with  $\alpha_1 = \alpha$  and  $\alpha_2 = \beta$ , we obtain  $(P_1)_e = G_e$  and  $\#(G_e/(P_2)_e) = 3$ . This means that  $(P_2)_e/(P_2)_e^o$  contains an element of order 2 of  $\Sigma_3 = G_e/G_e^o$ , which multiplies  $z \in \mathfrak{z}(\mathfrak{g}_e)$  by  $-1$ .

*Remark 5.5.* There are other ways to prove Proposition 5.3 if  $\mathfrak{g}$  is not of type  $G_2$ . Using Lemma 5.2 and information on  $\{\deg({}^e\mathcal{F}_i)\}$ , one can prove that  ${}^e\mathcal{F}_1, \dots, {}^e\mathcal{F}_l$  satisfy the hypotheses of Lemma 2.1 with  $A = G_e^o$ , which implies that the functions  ${}^e\mathcal{F}_1, \dots, {}^e\mathcal{F}_l$  freely generate the algebra  $\mathcal{S}(\mathfrak{g}_e)^{G_e^o}$  and hence  $\mathcal{S}(\mathfrak{g}_e)^{G_e^o} = \mathcal{S}(\mathfrak{g}_e)^{G_e}$ . There is also a way to describe  ${}^e\mathcal{F}_l$  almost explicitly. The intersection of  $e + \mathfrak{g}_f$  with the nullcone in  $\mathfrak{g}$  is isomorphic to a hypersurface in a 3-dimensional affine space with a unique singular point, a Klenian singularity [23]. Modulo the ideal  $({}^e\mathcal{F}_1, \dots, {}^e\mathcal{F}_{l-1}) \triangleleft \mathcal{S}(\mathfrak{g}_e)$ , the polynomial  ${}^e\mathcal{F}_l$  is the degree 2 part of the well-known equation defining that hypersurface. This statement can be deduced from [20, Section 7].

**Theorem 5.6.** *Let  $q$  be a subregular contraction of  $\mathfrak{g}$  and  $\mathcal{F}_1, \dots, \mathcal{F}_l$  the basic invariants in  $\mathcal{S}(\mathfrak{g})^G$ . Then  $\mathcal{F}_1^\bullet, \dots, \mathcal{F}_l^\bullet$  freely generate  $\mathcal{S}(q)^Q$  and satisfy the Kostant equality in  $q$ .*

*Proof.* This readily follows from Proposition 5.3 and Theorem 3.7. □

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